

Noise, diffusion, and hyperuniformity

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We consider driven many-particle models which have a phase transition between an active and an absorbing phase. Like previously studied models, we have particle conservation, but here we introduce an additional symmetry - when two particles interact, we give them stochastic kicks which conserve center of mass. We find that the density fluctuations in the active phase decay in the fastest manner possible for a disordered isotropic system, and we present arguments that the large scale fluctuations are determined by a competition between a noise term which generates fluctuations, and a deterministic term which reduces them. Our results may be relevant to shear experiments and may further the understanding of hyperuniformity which occurs at the critical point.

Among the remarkable behavior exhibited by driven many-body systems is a non-equilibrium phase transition between phases with different dynamics. For one interesting class of systems, the transition is from an active phase, where the system evolves in time forever, to an absorbing phase, where the dynamics eventually ceases. Such behavior has been studied theoretically[1, 2], as well as in experiments on sheared particles[3–6]. In all cases, the phase transition is effected by changing a control parameter such as the density.

The theoretical models showing this behavior consist of particles, each of which may be either static or active, depending on its local neighborhood. For example, in the random organization (RandOrg) model, initially introduced in [4] to model experiments on periodically sheared Brownian particles[3], particles of unit radius are placed in a volume. If two particles overlap, they are considered active. At every time step, each active particle is given a random displacement, or “kick”, while the isolated inactive particles remain in place. After an initial transient, and depending on the value of the control parameter ϕ (here, the density) of the model, the system evolves either into an absorbing state consisting only of isolated particles, or into an active phase where a well-defined fraction of the particles are active, and undergo unceasing random-like motion. This model has been shown[7] to belong to a larger class of absorbing phase transitions called the Manna universality class[8].

Although the system is out of equilibrium, the transition between active and absorbing phases has properties of a continuous phase transition, with characteristic critical exponents[8] and a well-defined critical value, ϕ_c , of the control parameter. Thus, the fraction of active particles grows as a power-law $\rho_a \propto |\phi - \phi_c|^\beta$, the time scale to reach an absorbing state (or the active steady state for $\phi > \phi_c$) diverges as $\tau \propto |\phi - \phi_c|^{-\nu_{||}}$, and an appropriately defined correlation length[9] diverges as $\xi \propto |\phi - \phi_c|^{-\nu_{\perp}}$.

Remarkably, however, unlike the equilibrium scenario where fluctuations diverge at the critical point, density fluctuations in these models are anomalously *small* at criticality[10], a phenomenon termed

hyperuniformity[11]. This is seen by measuring, as a function of ℓ , the density variance, $\sigma^2(\ell) \equiv \langle \delta\rho^2(\ell) \rangle$ in a volume ℓ^d . Asymptotically, $\sigma^2(\ell) \sim \ell^{-\lambda}$, with the exponent λ characterizing the magnitude of density fluctuations. For hyperuniform fluctuations, $\lambda > d$, so that density fluctuations decay much faster than for a random distribution, for which $\lambda = d$. The largest possible value of the exponent λ is $d + 1$, which occurs for systems like a periodic lattice [12]. An equivalent measure of hyperuniformity is that the structure factor $S(k)$ vanishes as $k \rightarrow 0$, typically characterized by an exponent α : $S(k) \sim k^\alpha$ [13].

Although several absorbing state systems have been shown to exhibit hyperuniformity, the underlying reasons for this unusual behavior have as yet to be elucidated. In this Letter, we examine a new class of models with an additional conserved quantity in the dynamics, which changes behavior in the active phase in an essential way. We then propose and solve a simple model analytically, and derive a novel Langevin equation for the coarse-grained dynamics. This leads us to propose that *large scale hyperuniform fluctuations are determined by a competition between a noise term which generates fluctuations, and a deterministic term which reduces them.* We suggest that this could provide a general mechanism leading to hyperuniformity.

The best-known classes of absorbing state models are the directed percolation (DP) [14] and the Manna class [15]. The difference is that in the Manna class, particle number is conserved, while in the DP class it is not. Here we study the effect of an additional conservation law, where the center of mass (COM) of two interacting particles is conserved by the dynamics, and find that it profoundly changes the behavior of the system. In colloidal systems at low densities, COM conservation may emerge naturally at low Reynolds number: if two spherical particles interact via a repulsive radial force when they are close, the dynamics will separate them along the line joining their centers. If more than two particles interact this is no longer true, but at low densities we expect this to be rare.

This additional conservation law does not seriously

modify the absorbing phase of the models, but it has a great effect on the active phase. In the absence of COM conservation, the active phase of models of the Manna class is characterized by $S(k) \sim k$ for intermediate values of k , going over to a constant as $k \rightarrow 0$ [16], indicating that the system in the active phase is not hyperuniform. The addition of COM conservation changes this materially, with $S(k) \sim k^2$ as $k \rightarrow 0$ in the active phase. This behavior corresponds to hyperuniform density fluctuations going as $\sigma^2(\ell) \sim \ell^{-(d+1)}$, which is the fastest decay possible, on par with that of a perfect lattice.

After presenting numerical results on two models, we will introduce and study analytically a simple one-dimensional model, for which we derive a Langevin equation valid for large densities and long length scales. While the average profile (averaged over realizations of the dynamics) obeys a diffusion equation, we find that the additional conservation law modifies the fashion that noise enters in an essential way. We find that hyperuniformity emerges through a competition between diffusion of the average density profile, which reduces density fluctuations, and noise emerging from the stochastic dynamics, which creates them.

We first present numerical results on two variants of models found in the literature, each modified to include COM conservation [17]:

I) Random Organization (2d): Particles of radius a are placed randomly in an $L \times L$ box. If two particles overlap, they are considered active and are given a random displacement; otherwise they are static. We modify the model of [4] such that the displacement of a pair of active particles is along the axis connecting their centers, with an amplitude chosen from a uniform distribution in the range $[0, 2a]$; see Fig. 1(a).

II) Manna Model (2d): This model [1] is defined on a square lattice where each site may hold any number of particles; sites with more than two particles are considered active. Two particles on an active site move in opposite directions to adjacent neighboring site, with vertical or horizontal moves chosen with equal probability.

These models exhibit a phase transition between a low density absorbing phase and a high density active phase, similar to their forbears without COM conservation. Moreover, the decay of the density fluctuations at the critical point is the same as it is for the ordinary Manna-class models. However, in contrast to the ordinary Manna class, fluctuations in the active phase of both of these models is hyperuniform.

Hyperuniformity is readily seen by measuring the structure factor $S(k) = \frac{1}{N} \left| \sum_{i=1}^N \exp(-ikr_i) \right|^2$. Since $S(k \rightarrow 0) = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}$ for an infinite system [11], hyperuniformity at large scales implies that $S(k \rightarrow 0) \rightarrow 0$ [18][11, 19], typically [20] as a power of k : $S(k) \propto k^\gamma$. When $\gamma < 1$ the density fluctuations are given by $\sigma^2(\ell) \propto \ell^{-\gamma-d}$, while if $\gamma > 1$ then the fluctuations are

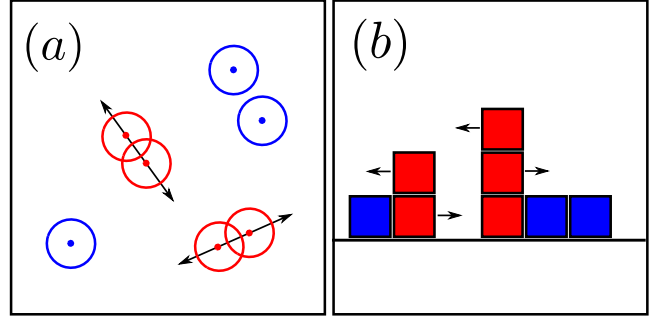


Figure 1. (a) An illustration of the random organization model with center of mass conserving dynamics. (b) The one dimensional toy model. In both cases blue particles are inactive and red are active.

what might be termed ‘maximally hyperuniform’, with $\sigma^2(\ell) \propto \ell^{-d-1}$ [13].

As seen in Figures 2 and 3, which show $S(k)$ above the critical density for the two models described above, as $k \rightarrow 0$, $S(k) \propto k^2$, implying that the systems are maximally hyperuniform. As the density approaches the critical point, a crossover occurs from small values of k , where $S(k) \propto k^2$, to larger k , where $S(k) \propto k^{0.45}$. We denote the crossover wavevector by k_+ , which is most clearly seen for $\rho = 1.7625$ in Figure 3. The large k regime is the same as the critical behavior of absorbing phase transitions without the COM symmetry, as seen in Reference [10]. As the density approaches its critical value from above, the crossover point k_+ approaches 0, suggesting a diverging correlation length defined as $\xi = 2\pi/k_+$.

Thus, the central effect of COM conservation is that for all values of density $\rho > \rho_c$ the system becomes hyperuniform, so that $\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} \rightarrow 0$ in an infinite system. This is in contrast to the case where this symmetry is absent and hyperuniformity occurs only when the system is tuned to the critical point. Figure 4 shows this same behavior in real space for the two dimensional Manna model. Near the critical point, the system shows the usual scaling $\sigma^2(\ell) \propto \ell^{-2.45}$ [10], while for large densities, $\sigma^2(\ell) \propto \ell^{-3}$ similar to the fluctuations of a crystal whose ‘atoms’ are randomly displaced from their lattice sites.

In order to better understand the origins of the observed hyperuniformity, we now introduce a one-dimensional model, which we call the COMCon model, which admits analytic analysis in its active phase. In this model, a site i may have any number n_i of particles. If $n_i > 1$, then at a rate $\omega_0 \cdot (n_i - 1)$, two particles move from site i - one moves to site $i - 1$ while the second moves to $i + 1$, thus conserving the center of mass. The phase transition for the COMCon model is of a different character from that of models I and II, as seen in both the structure factor, which diverges as $S(k) \sim k^{-0.5}$ as $\rho_c \approx 1$ is approached from below, and in the fraction of active particles, which jumps as ρ crosses ρ_c . However, like these other two models, $S(k) \sim k^2$ in the active

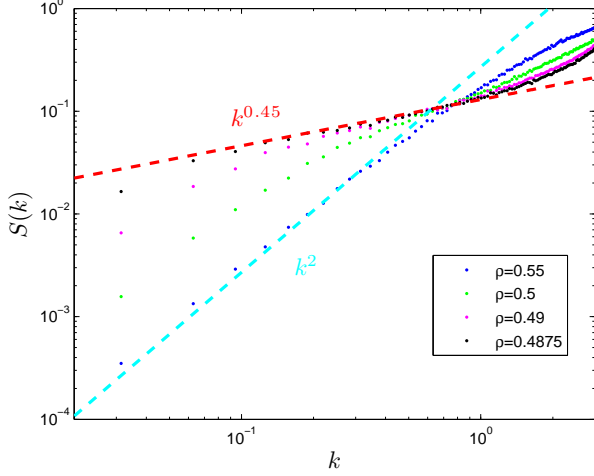


Figure 2. 2D RandOrg with COM conservation: The structure factor for $\rho > \rho_c$ scales as k^2 as $k \rightarrow 0$. Here $L = 400$, and the number of realizations is 50. At the critical point, $\rho_c \approx 0.487$, the usual scaling $k^{0.45}$ is seen.

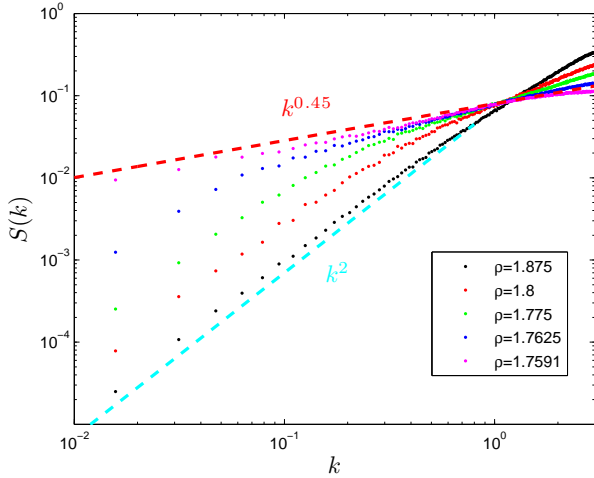


Figure 3. 2D Manna model with COM conservation for $\rho > \rho_c$. Here $L = 400$, and the number of realizations is 50. Here $\rho_c \approx 1.7591$.

phase of COMCon, as seen in Figure 5. Here we will only study the behavior of the model where $n_i \gg 1$, which ensures that the system is everywhere active.

Let us begin by studying the coarse grained dynamics of COMCon in the continuum limit at large densities $\rho > \rho_c \approx 1$. As shown in the Supplementary Materials, the Langevin equation for the continuum density $n(x, t)$ has a diffusive term and a noise term emanating from the microscopic random dynamics:

$$\partial_t n = D \partial_{xx} n + A \partial_{xx} (\sqrt{n-1} \eta) \quad (1)$$

Here $\langle \eta(x, t) \eta(x', t') \rangle = \delta(x - x') \delta(t - t')$, $A = \sqrt{\omega_0} a^2$, and $D = \omega_0 a^2$ are constants which depend on the microscopic rates, and a is the lattice constant. The key

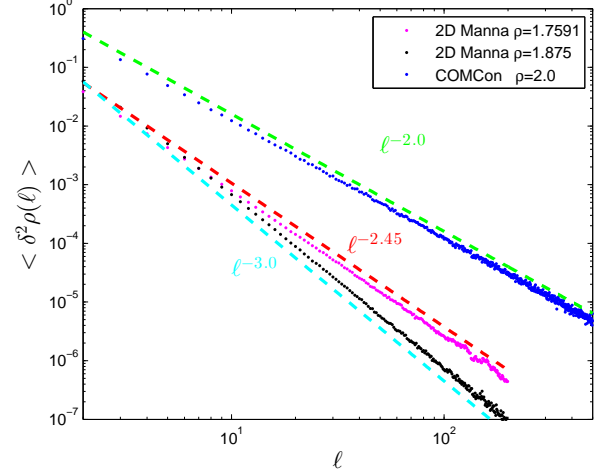


Figure 4. Real space fluctuations of the 2D COM Manna model and the COMCon model. In both cases the density fluctuations at high density is ‘maximally hyperuniform’, $\sigma^2(\ell) \propto \ell^{-d-1}$. Near the critical point of the Manna model ($\rho_c \approx 1.7591$) $\sigma^2(\ell) \propto \ell^{-2.45}$, similarly to the case where COM symmetry is absent.

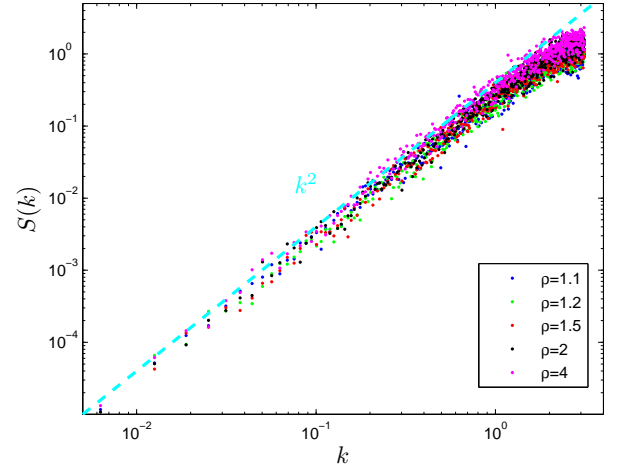


Figure 5. The structure factor, $S(k) \propto k^2$, of COMCon for density $\rho > 1$, corresponding to the active phase. The system size here is $L = 1000$ and the number of realizations is 20 for each density.

point here is that due to particle conservation *and* COM conservation, the noise term comes in as a second spatial derivative. It is this that leads to the far-reaching consequence of hyperuniformity in the active phase, as discussed below.

We note that Equation 1 only describes the model when the average density is large, since when $n < 1$ we get spurious results - complex noise and diffusion - which is not present in the discrete model. Such issues arise because we chose the transition rates to be $\omega_0(n-1)$, and treat n as a continuous variable. To avoid this anomalous

behavior, we will restrict our analysis to the high density regime [21], that is, the active phase.

The second point to note is that if only particles were conserved, the noise term would have a single space derivative, familiar from the Model B dynamics[22] of ordinary diffusion. Such a noise term would not lead to hyperuniform fluctuations.

The easiest way to see that the density fluctuations deriving from Equation 1 are hyperuniform is to compute the structure factor $S(k) = \frac{1}{N} \langle n_k n_{-k} \rangle$, where $n_k = \int dx n(x) e^{-ikx}$. Writing $n(x) = n_0 + \delta n$, where n_0 is the average density, and expanding to first order in δn , the last term in Equation 1 can be approximated by $A\sqrt{n_0-1} \partial_{xx} \eta$. Writing the Langevin in Fourier space, we get

$$\partial_t \delta n_k = -k^2 D \delta n_k - A\sqrt{n_0-1} k^2 \eta_k \quad (2)$$

whose solution is

$$\delta n_k = C e^{-Dk^2 t} + k^2 A\sqrt{n_0-1} e^{-Dk^2 t} \int_0^t dt' e^{Dk^2 t'} \eta_k \quad (3)$$

where C is set by the initial condition. We may now compute the k dependence of the structure factor, by using the fact that $S(k) = \frac{1}{N} \langle \delta n_k \delta n_{-k} \rangle$:

$$S(k) = C^2 e^{-2Dk^2 t} + \frac{n_0-1}{n_0} \frac{A^2 k^2}{2D} (1 - e^{-2Dk^2 t}) \quad (4)$$

$$\xrightarrow[t \rightarrow \infty]{} \frac{n_0-1}{n_0} \frac{A^2}{2D} k^2. \quad (5)$$

Thus, the result of COM conservation is that $S(k) \propto k^2$.

This model calculation points up one way that hyperuniformity can occur. In particular, we note that the two terms of Equation 2 have competing roles. The diffusion term reduces fluctuations so that a given mode with wave vector k decays with a rate $-k^2 D$ for small k . On the other hand the second term on the right hand side generates fluctuations which scale as $k^2 \eta_k$, which are greatly suppressed at small k values.

In conclusion we have shown that COM leads to hyperuniformity emerging as a competition between a diffusion term suppressing a fluctuations and a noise term which generates fluctuations. The interplay between the scaling with k of these two terms determines the steady state fluctuations. It is interesting to speculate that other hyperuniform systems are governed by a similar Langevin equation, perhaps with different scalings. For example, for a Langevin equation (expressed in Fourier space) of the form

$$\partial_t n_k = -D|k|^\alpha n_k - A k^\beta \eta_k \quad (6)$$

the structure factor scales as $S(k) = |k|^{2\beta-\alpha}$. Typically, if the noise has no special properties, $\beta = 1$, and in order

for hyperuniformity to occur we would need $\alpha < 2$. The exponent α determines the time it takes the system to reach steady state, which corresponds to the decay time of the longest mode $k = \frac{2\pi}{L}$. From Equation 6 it is found that that $\tau \propto L^\alpha$ so that if $\alpha < 2$ a steady state is reached much faster than diffusion where $\tau \propto L^2$. Faster than diffusion scaling is found at the critical point of the Manna universality class [2] for dimensions smaller than four which have been shown to be hyperuniform [10].

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Appendix A: Direct derivation of Langevin equation

In this section we derive the coarse grained Langevin equation for the COMCon lattice model in one dimension. Towards this goal we first show that center-of-mass (COM) conservation law constrains the current to be a divergence of a field. Particle conservation implies that the change in the density can be written as a divergence of the current:

$$\partial_t n = -\nabla \cdot J. \quad (7)$$

Thus, by the divergence theorem, the change in the particle number within any volume must come from the boundary.

As a result of COM conservation each dimension can be associated with global conserved quantities $R_\alpha = \int d^d r n(r) r_\alpha$, so that $\partial_t R_\alpha = 0$, where the integral is over all space. If the integral is taken over a finite portion of the system, any change in R_α is due to particles entering or exiting along the surface enclosing this region. This can be expressed as $\partial_t R_\alpha = -\int dS \cdot J_{R,\alpha}$ where $J_{R,\alpha}$ is the current associated with the COM conservation. Using Eq. 7 the change in R_α can be computed,

$$\partial_t R_\alpha = \int d^d r r_\alpha \partial_t n(r) \quad (8)$$

$$= -\int d^d r r_\alpha \nabla \cdot J \quad (9)$$

$$= -\int dS \cdot (J r_\alpha) + \int d^d r J_\alpha \quad (10)$$

where in the last line we integrated by parts. COM conservation requires the last term to also scale as the surface, implying that J_α can be written as the divergence of a vector, $J_\alpha = -\nabla \cdot \sigma_\alpha$. Together with Equation 7, this gives

$$\partial_t n = \nabla \cdot \nabla \cdot \sigma_\alpha. \quad (11)$$

In one dimension, σ may be written $\sigma = f(n) + g(n)\eta$, being composed of a deterministic term $f(n)$ and a term $g(n)\eta$ which accounts for the stochastic motion, where η is assumed to be zero averaged white noise $\langle \eta(x, t) \eta(x', t') \rangle = \delta(x - x') \delta(t - t')$.

Since $\langle \eta \rangle = 0$, the term $f(n)$ can be found by computing $\langle \Delta n \rangle = \int_0^{\Delta t} dt \langle \partial_t n \rangle = \partial_{xx} f(n)$. The noise term $g(n)$ can be found by looking at $\langle (\Delta n)^2 \rangle$, since in an infinitesimal duration, the noise term $\int_0^{\Delta t} dt g(n) \eta \sim \sqrt{\Delta t}$, which dominates over the deterministic term which scales as Δt . For the COMCon model, two particles exit a site at a rate $\omega_0(n_i - 1)$, and move in opposite directions to adjacent sites. The average change in the occupancy $\langle \Delta n_i \rangle$ is composed of three terms, where the first is due to the aforementioned transition and the other two account for transitions in neighboring sites. Hence,

$$\begin{aligned} \langle \Delta n_i \rangle &= [-2(n_i - 1) + (n_{i+1} - 1) + (n_{i-1} - 1)] \omega_0 \Delta t \\ &\approx \omega_0 a^2 \Delta t \frac{\partial^2 n}{\partial^2 x} \end{aligned} \quad (12)$$

where we took the continuum limit by assuming that sites are separated by a lattice constant a . Therefore, $f(n) = \omega_0 a^2 n$. Eq. 12 implies that the average density profile evolves via usual diffusion, with no alterations due to COM conservation.

To find $g(n)$ we compute $\langle (\Delta n)^2 \rangle$ since as noted, the fluctuations dominate, allowing us to neglect any deterministic terms. This is calculated from the term $\partial_{xx} g(n) \eta$. For simplicity, we discretize space, taking the lattice constant to be a :

$$\Delta n_i = \int_0^{\Delta t} dt \frac{1}{a^2} [g(n_{i+1}) \eta_{i+1} + g(n_{i-1}) \eta_{i-1} - 2g(n_i) \eta_i]. \quad (13)$$

Assuming that $\langle \eta_i \eta_j \rangle = \delta_{ij}$, the second moment is then given by

$$\langle (\Delta n_i)^2 \rangle = \frac{1}{a^4} [g^2(n_{i+1}) + g^2(n_{i-1}) + 4g^2(n_i)] \Delta t. \quad (14)$$

This is compared to the exact result for the COMCon model to order $O(\Delta t)$,

$$\langle (\Delta n_i)^2 \rangle = [(n_{i-1} - 1) + (n_{i+1} - 1) + 4(n_i - 1)] \omega_0 \Delta t, \quad (15)$$

which implies that $g(n) = \sqrt{\omega_0 a^4} \sqrt{n-1}$. The main result of this section is the Langevin equation, given by,

$$\partial_t n = \omega_0 a^2 \partial_{xx} n + \sqrt{\omega_0 a^4} \partial_{xx} \sqrt{n-1} \eta \quad (16)$$

Appendix B: Field theory derivation of Langevin equation

In this section we present an alternative derivation of the coarse grained Langevin equation for the COMCon model. Towards this goal, we employ the framework of Ref. [23] to compute its field theory and then take the (spatial) continuum limit. Let $n_i(t)$ denote the number of particles at site i and at time t which is discretized into intervals of dt . We first compute the probability measure $P(\{n_i(t)\})$ of a given trajectory $\{n_i(t)\}$, which we express as a functional integral over an auxiliary field $\{p_i(t)\}$; this will allow us to identify the action $\mathfrak{s}(\{n_i\}, \{p_i\})$:

$$P(\{n_i(t)\}) = \frac{1}{Z} \int \mathcal{D}p e^{-\mathfrak{s}(\{n_i\}, \{p_i\})} \quad (17)$$

where $\mathcal{D}p \equiv \prod dp_j(t)$ and the product over all lattice sites and over all times in the interval of interest. The action can be identified by expressing the average of an arbitrary functional $\langle O[\{n_i(t)\}] \rangle$ over some period of time as a path integral over all possible trajectories, where the dynamics of the allowed trajectories is enforced using delta functions. We write these dynamics as $\Delta n_i(t) \equiv n_i(t + dt) - n_i(t) = J_i$, where the ‘currents’ J_i reflect the occupancy changes permitted by the rules, and then average over all values of J_i :

$$\langle O \rangle = \frac{1}{Z} \left\langle \int \mathcal{D}n O[\{n_i(t)\}] \Pi_i \delta(\Delta n_i(t) - J_i) \right\rangle_J \quad (18)$$

where Π_i represents a product over all independent currents and times.

For the COMCon, the COM conservation condition gives $J_{i-1} = J_{i+1} = 1$ and $J_i = -2$. Using the Martin-Siggia-Rose (MSR) procedure [24–26], the delta functions are written as integrals over plane waves:

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}n \mathcal{D}p O[\{n_i(t)\}] \left\langle \Pi_t e^{-\sum_i p_i (\Delta n_i(t) - J_i)} \right\rangle_J \quad (19)$$

where Π_t is a product over discretized time. Hence the probability density of a given trajectory is

$$P(\{n_i(t)\}) = \frac{1}{Z} \int \mathcal{D}p \left\langle \Pi_t e^{-\sum_i p_i (\Delta n_i(t) - J_i)} \right\rangle_J \quad (20)$$

where we identify the action \mathfrak{s} by comparing Equations 17 and 20. In this expression, the only model-dependent quantity is the generating function, $\langle e^{\sum_i p_i J_i} \rangle$. For the COMCon model,

$$\langle e^{\sum_i p_i J_i} \rangle = \left(1 - \omega_0 \sum_i (n_i - 1) dt \right) \quad (21)$$

$$+ \omega_0 \sum_i (n_i - 1) dt e^{\Delta^2 p_i} \quad (22)$$

$$\approx e^{\omega_0 \sum_i (n_i - 1) dt (e^{\Delta^2 p_i} - 1)} \quad (23)$$

where $\Delta^2 p_i \equiv p_{i+1} - 2p_i + p_{i-1}$. The first term is the probability that no transition occurs during time dt , and the second term sums over the probabilities of a single transition. The probability of two transitions occurring is of order $(dt)^2$ and is therefore neglected. Reverting to continuum time by taking $\sum dt \rightarrow \int dt$ which leads to the final action \mathfrak{s} given by:

$$\mathfrak{s} = \sum_i \int dt p_i \partial_t n_i - \omega_0 (n_i - 1) \left(e^{\Delta^2 p_i} - 1 \right). \quad (24)$$

To obtain the continuum limit we assume that adjacent sites are separated by a small distance a and expand the second term, so that $p_{i\pm 1} \approx p_i \pm a \partial_x p_i + \frac{1}{2} a^2 \partial_{xx} p_i \pm \frac{a^3}{3!} \partial_x^{(3)} p_i + \frac{a^4}{4!} \partial_x^{(4)} p_i$:

$$e^{\Delta^2 p_i} - 1 \approx e^{a^2 \partial_{xx} p + 2a^4/4! \partial_x^4 p_i} - 1 \quad (25)$$

$$\approx a^2 \partial_{xx} p + \frac{2a^4}{4!} \partial_x^4 p + \frac{1}{2} a^4 (\partial_{xx} p)^2 \quad (26)$$

The second term in Eq. 26 subdominant since it has more spatial derivatives than the first term and henceforth neglected. The continuum action is then given by:

$$\mathfrak{s} = - \int dt \int dx [p \partial_t n - \omega_0 a^2 p \partial_{xx} n - \omega_0 (n - 1) \frac{1}{2} a^4 (\partial_{xx} p)^2] \quad (27)$$

where the second term is obtained through integration by parts. The Langevin equation associated with this action is identified by a writing a trial Langevin equation with different possible terms and then performing the MSR procedure to obtain the resulting action. Motivated by Section we assume

$$\partial_t n = D \partial_{xx} n + \partial_{xx} \xi$$

where $\xi(x, t)$ is a Gaussian noise term and its dependence on n_i is to be determined. To obtain the action we express the average of any quantity $O[\{n_i(t)\}]$ as a path integral over all possible trajectories, where the dynamics are enforced by delta functions. Subsequently, the delta function written in terms of the auxiliary field p ,

$$P\langle O \rangle = \frac{1}{Z} \left\langle \int \mathcal{D}n O[\{n_i(t)\}] \delta(\partial_t n - D \partial_{xx} n + \partial_{xx} \xi) \right\rangle_\xi \\ = \frac{1}{Z} \left\langle \int \mathcal{D}n \mathcal{D}p O[\{n_i(t)\}] e^{-\int dt \int dx p (\partial_t n - D \partial_{xx} n - \partial_{xx} \xi)} \right\rangle_\xi.$$

The first two terms are identical those in Eq. 27, implying that the diffusion constant $D = \omega_0 a^2$. The angular brackets denote an average over the Gaussian noise ξ , with distribution $p(\{\xi_i\}) \propto e^{-\int dx \int dt \frac{\xi^2}{2g^2(n)}}$ where we $g(n)$ is to be found. Only the last term depends on ξ and its average can be computed by discretizing space and time and then performing the Gaussian integral over ξ in a period of dt . This yields,

$$\left\langle e^{-\int dx dt A p \partial_{xx} \xi} \right\rangle_\xi = \quad (28)$$

$$= \int \Pi_i d\xi_i \frac{1}{\sqrt{2\pi g^2(n_i)}} e^{-\sum_i dt \frac{A}{a^2} p_i \Delta^2 \xi_i - \sum_i \frac{\xi_i^2}{2g^2(n_i)} dt} \quad (29)$$

$$= \exp \left[\sum_i \frac{1}{2} \left(\frac{A}{a^2} \Delta^2 p_i \right)^2 g^2(n_i) dt \right] \quad (30)$$

$$\approx \exp \left[\int dx dt \frac{1}{2} \left(\frac{1}{a^2} \partial_{xx} p \right)^2 g^2(n) \right] \quad (31)$$

where the continuum limit has been taken in the last line. Here, $\Delta^2 \xi_i = \xi_{i+1} - 2\xi_i + \xi_{i-1}$ is the second (spatial) difference of ξ , and likewise $\Delta^2 p_i$. Comparing to Eq. 27 we see that $g^2(n) = \omega_0 a^4 (n - 1)$ in agreement with Eq. 16.

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